Explicit Modeling of Singularities in Deformable Registration

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Motivation

- Many image analysis tasks require deformable registration between images or volumes e.g. in medical imaging and computer vision.
- To ensure plausible deformations models are often restricted to smooth transformations with smooth inverses (**diffeomorphisms**).
- In reality not everything behaves smoothly everywhere so these models may fail to capture essential details.
- We propose a framework for explicitly modeling certain **topological changes** in a piecewise-diffeomorphic setting.

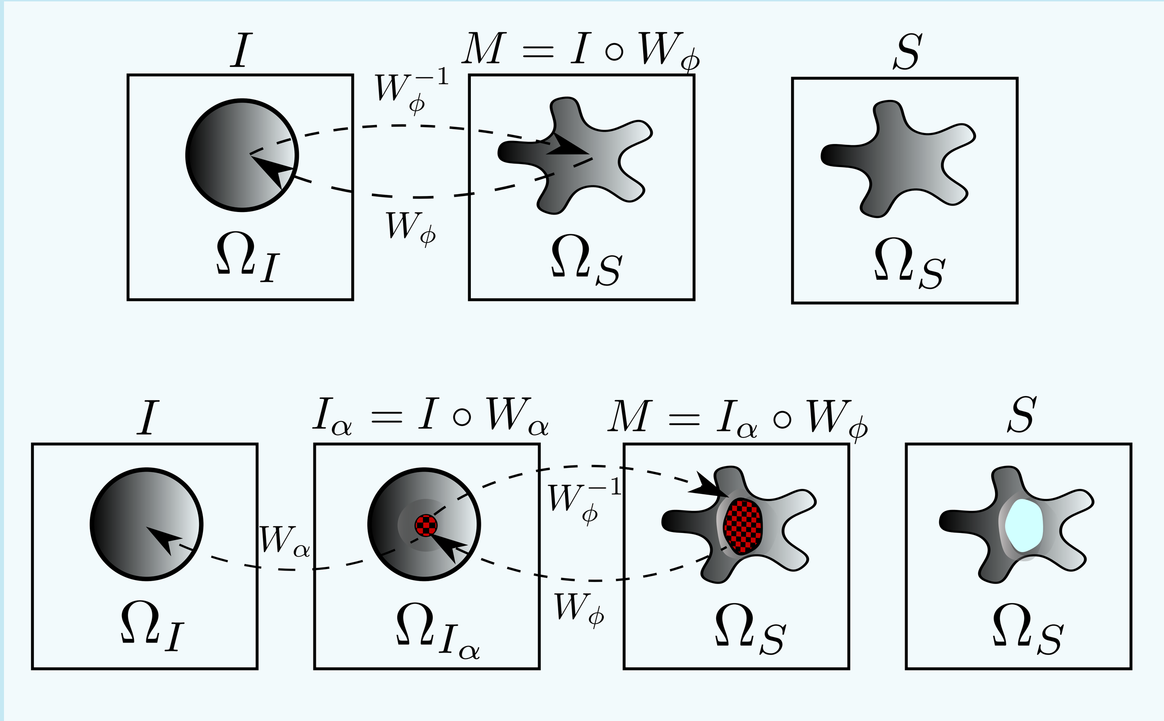

Figure 1 Diffeomorphic subjects are registered by a diffeomorphism W_ϕ (top). Non-diffeomorphic subjects are registered using the composition of an initial non-diffeomorphic transformation W_α followed by a diffeomorphism W_ϕ (bottom). The checkered area represents undefined matter grown from an infinitesimal point. Multiple points at the periphery of the teared hole in I_α correspond to the same point at the center of the tear in I so W_α is not bijective.

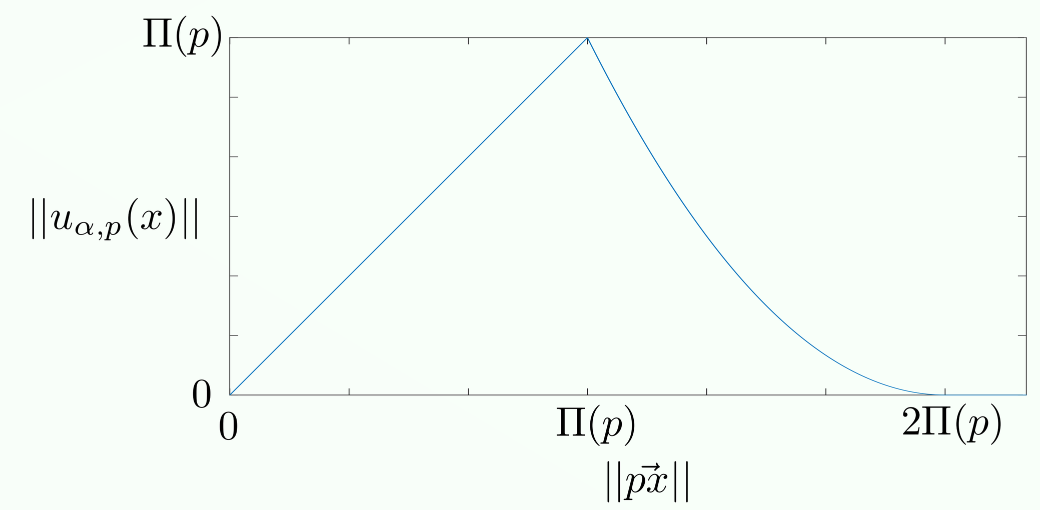

Figure 2 Magnitude of displacement as a function of distance from endpoint perspective.

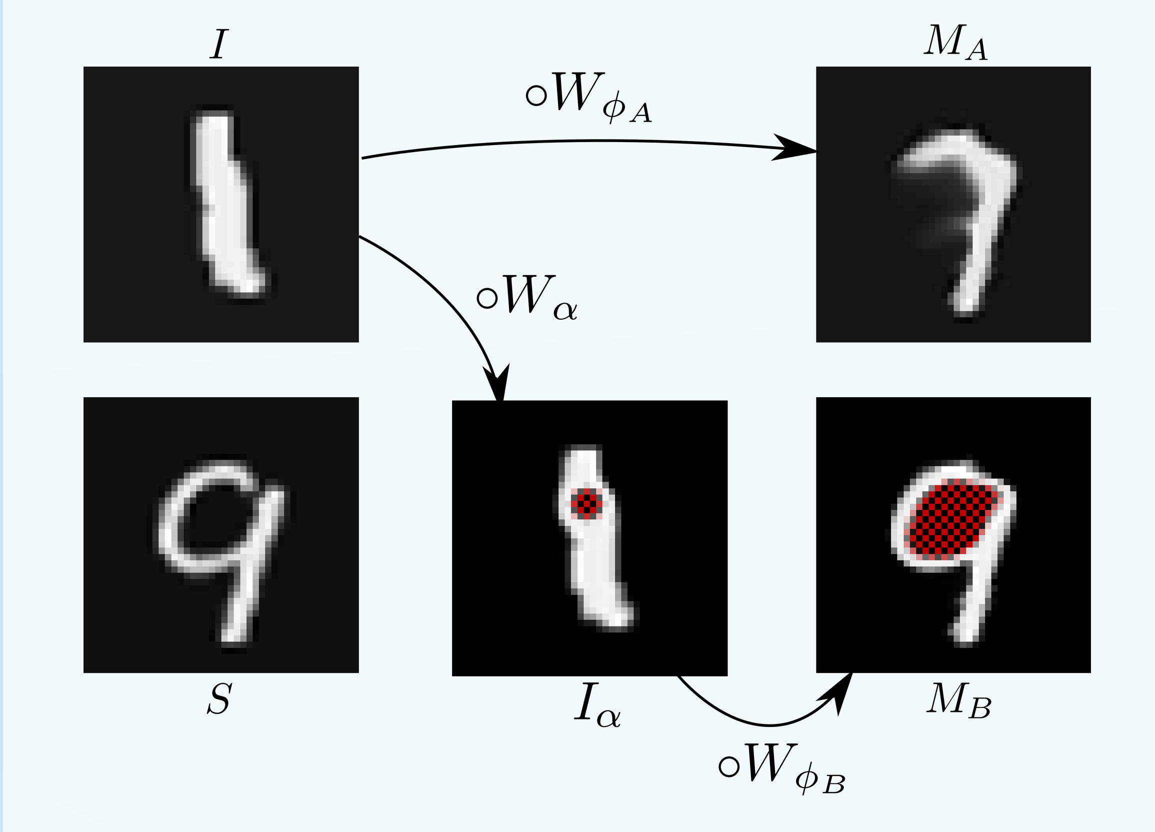

Figure 3 Registration of MNIST images with and without allowing topological change. A purely diffeomorphic model fails to produce a desirable result (upper). Introducing a topological change allows the model to find a better match (bottom). In this case the center of the change is preseeded.

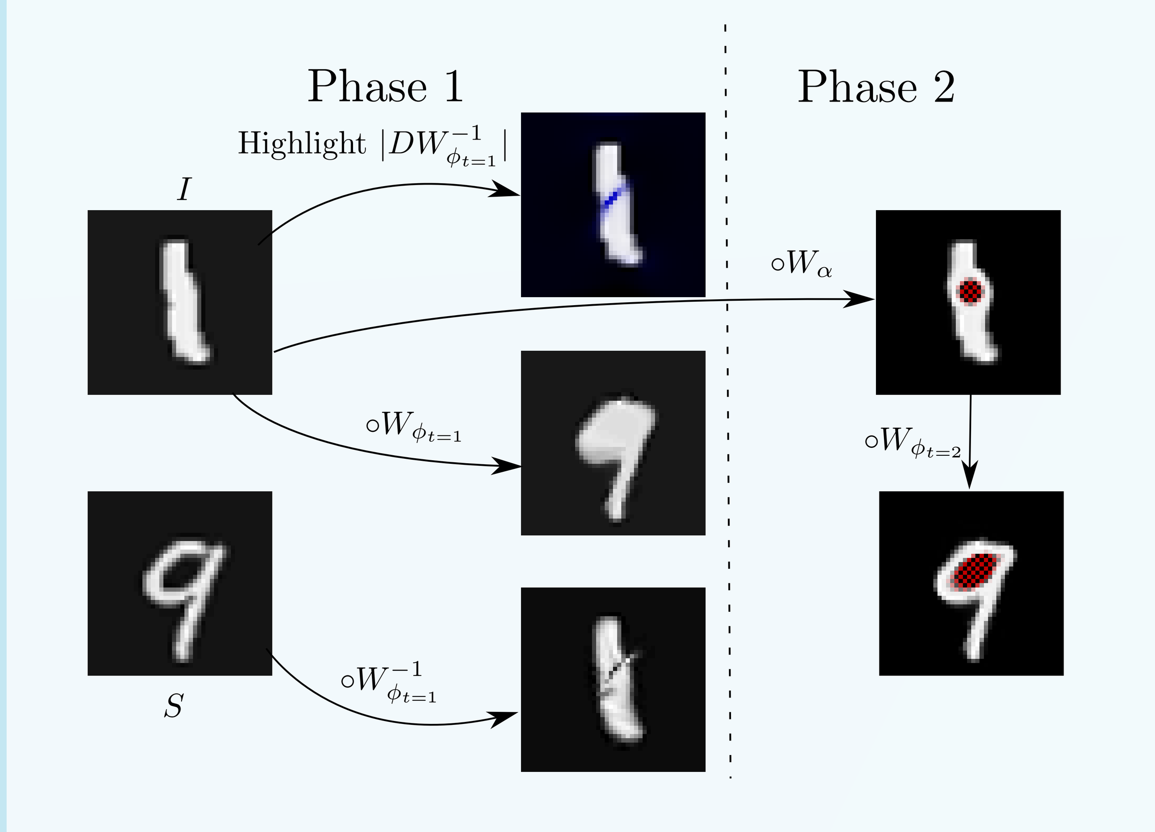

Figure 4 An initial diffeomorphic registration squeezes the hole in the "9" in order to match the "1". This can be measured in the Jacobian and, since the match is still bad in the center of the squeeze, this hints of a topological difference. Seeding a topological change at this center allows the model to find a desirable match.

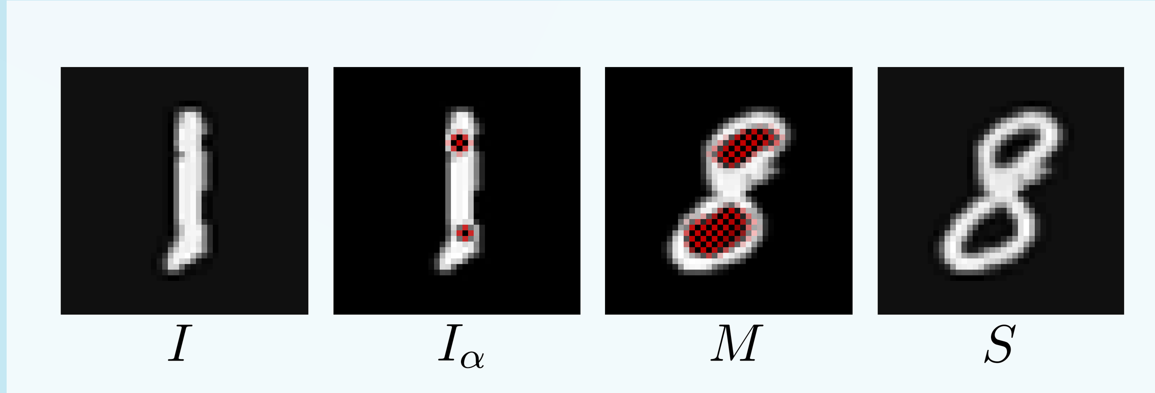

Figure 5 Registration with multiple preseeded topological changes.

Deformable registration

In deformable registration we seek the transformation

$$W\colon \Omega_S \to \Omega_I \tag{1}$$

that aligns an input image $I\colon \Omega_I \to \mathbb{R}^d$ into the coordinate system of stationary image $S\colon \Omega_S \to \mathbb{R}^d$. The deformed image is given by

$$M = I \circ W \tag{2}$$

and the deformation is given by

$$W(x) = x - u(x) = y \tag{3}$$

where $u\colon \mathbb{R}^n \to \mathbb{R}^n$ is the displacement vector field representing the change of coordinates from the endpoint perspective. If W^{-1} exists then correspondingly

$$W^{-1}(y) = y + u(x) = x. \tag{4}$$

We then search for W minimizing some error- and regularity terms.

In diffeomorphic registration W is often denoted by ϕ but here we use ϕ as a subscript to signify diffeomorphicity of W.

Deformation Composition

We define the transformation W to be given by the composition

$$W = W_\alpha \circ W_\phi \tag{5}$$

of the non-diffeomorphic transformation $W_\alpha(x) = x - u_\alpha(x)$ and diffeomorphic transformation $W_\phi(x) = x - u_\phi(x)$. The model controlling the former is designed to achieve diffeomorphicity by "tearing holes" in the subject while the latter may be controlled by any diffeomorphic deformation framework.

Tearing the subject apart leaves *void* area or volume which must be reflected in the matching term. We model this using an alpha channel $\alpha\colon \Omega_S \to \{0,1\}$ and define an L^2-like term

$$L^2_\alpha(S, I, W, \alpha) = \int_{\Omega_S} \alpha(x)^2 (M(x) - S(x))^2 \, dx \tag{6}$$

$$= \int_{\Omega_S} (M_\alpha(x) - S(x))^2 \, dx \tag{7}$$

$$\text{where } M_\alpha(x) = \alpha(x) M(x) + (1 - \alpha(x)) S(x). \tag{8}$$

Growing Spheres

Let $\Pi\colon \Omega_{I_\alpha} \to \mathbb{R}_{\geq 0}$ such that $\Pi(p)$ is the radius of a sphere centered at $p \in \Omega_\alpha$. A point $x \in \Omega_{I_\alpha}$ is considered void if for any p

$$||\vec{px}|| < \Pi(p). \tag{9}$$

A point $x \in \Omega_S$ is then considered void if for any p

$$||\vec{pW_\phi}(x)|| < \Pi(p) \tag{10}$$

The expansion of a sphere radially displaces mass surrounding p. A point $x \in \Omega_{I_\alpha}$ within the sphere correspond to its center so the endpoint displacement is simply

$$u_{\alpha,p}(x) = \Pi(p) \frac{\vec{px}}{||\vec{px}||} = \vec{px}. \tag{11}$$

Outside of the sphere the displacement magnitude decreases smoothly with a Wendland function

$$u_{\alpha,p}(x) = \Pi(p) \frac{\vec{px}}{||\vec{px}||} (1 - (||\vec{px}|| - \Pi(p))/\Pi(p))^2_+ . \tag{12}$$

The magnitude of the endpoint displacement as a function of distance is shown in figure 2. It is compactly supported so that it covers only x with $||\vec{px}|| < 2\Pi(p)$.

Future Work & Challenges

- Overlapping regions of influence
- Non-discrete singularities (e.g. lines in 2D or planes in 3D)
- Detection of and sensitivity towards locations of topological changes
- Application to real problems
- Unifying framework with other topological changes such as sliding motion